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# Best Error Bounds for Quartic Spline Interpolation

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## 1. INTRODUCTION

To represent a function that is not analytic, the most frequently used methods of piecewise polynomial approximation are piecewise linear interpolation and piecewise cubic interpolation. Piecewise linear interpolation has the advantage that it preserves monotonicity and convexity. Also, the maximum error between a function and its linear interpolant can be controlled directly by the spacing between the data points. Some of the main disadvantages of using piecewise linear functions are that such functions have corners where two linear pieces meet and that to achieve a prescribed accuracy usually requires far more data than are required by some higher order methods (i.e., cubic spline interpolation).

Various aspects of interpolatory cubic spline functions were considered by I. J. Schoenberg [9], Garrett Birkhoff and Carl DeBoor [2, 3], and A. Meir and A. Sharma [10]. Hall and Meyer [6] gave an interesting application of the theorem of Birkhoff and Priver [4] concerning optimal error bounds for complete cubic spline interpolation. In this paper we consider a related problem. For its description we denote by $S_{k,4}$ the class of quartic spline functions $s(x)$ such that

$$\begin{aligned} &\text{(i)} \quad s(x) \in C^2[0, 1], \\ &\text{(ii)} \quad s(x) \text{ is a quartic in each } [x_i, x_{i+1}], \qquad i = 0, 1, ..., k-1, \end{aligned} \tag{1.1}$$

where

$$0 = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = 1. \tag{1.2}$$

We now state our main results:

THEOREM 1. *Given arbitrary numbers* $f(x_i)$, $i = 0, 1, ..., k$, $f(z_i)$, $i =$



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$1, 2, ..., k$ $((2z_i = x_i + x_{i-1})$, $f'(x_0)$, $f'(x_k)$, *there exists a unique* $s(x) \in S_{k,4}$ *such that*

$$s(x_i) = f(x_i), \qquad i = 0, 1, ..., k, \; s'(x_0) = f'(x_0), \; s'(x_k) = f'(x_k); \qquad s(z_i) = f(z_i), \qquad i = 1, 2, ..., k. \tag{1.3}$$

THEOREM 2. *Let* $f \in C^5[0, 1]$. *Let* $s(x)$ *be the unique spline function satisfying* (1.1) *and* (1.3). *Then we have*

$$|f(x) - s(x)| \leqslant \frac{C_0 h^5}{5!} \max_{0 \leqslant x \leqslant 1} |f^{(5)}(x)|, \tag{1.4}$$

*where*

$$C_0 = \left(\frac{1}{30} + \frac{\sqrt{30}}{3}\right)\left(\frac{1}{4} - \frac{1}{\sqrt{30}}\right)^{1/2} = \max_{0 \leqslant t \leqslant 1} |c(t)| \tag{1.5}$$

$$c(t) = \frac{3t^2(1-2t)(1-t)^2 + t(1-t)(1-2t)}{6}. \tag{1.6}$$

*Also we have*

$$|f'(x_i) - s'(x_i)| \leqslant \frac{h^4}{6!} \max_{0 \leqslant x \leqslant 1} |f^{(5)}(x)|, \qquad i = 1, 2, ..., k-1. \tag{1.7}$$

*Furthermore,* $C_0$ *in* (1.5) *cannot be improved for an equally spaced partition. Inequality* (1.7) *is also best possible. Also we have*

$$|f'(x) - s'(x)| \leqslant c_1 \frac{h^4}{6!} \max_{0 \leqslant x \leqslant 1} |f^{(5)}(x)|. \tag{1.8}$$

*Remark.* It is of interest to note that both the complete cubic spline interpolation and the quartic spline interpolation of Theorem 1 belong to $C^2[0, 1]$. But in [2] they proved that there is an increase in the order of approximation of the first derivative at the knots when equal intervals are used. Our quartic compares very well in this respect (see (1.7) and (1.8)). See [7] also. For many interesting contributions about quadratic spline interpolation we refer to [1], [8].

## 2. PRELIMINARIES

If $q(z)$ is a quartic polynomial on $[0, 1]$, then it is easy to verify that

$$q(z) = q(0)\, P_1(z) + q(\tfrac{1}{2})\, P_2(z) + q(1)\, P_3(z) + q'(0)\, P_4(z) + q'(1)\, P_5(z), \tag{2.1}$$

where

$$
\begin{aligned}
P_1(z) &= 1 - 11z^2 + 18z^3 - 8z^4 = (1-2z)(1-z)^2(1+4z),\\
P_2(z) &= 16z^2 - 32z^3 + 16z^4 = 16z^2(1-z)^2,\\
P_3(z) &= -5z^2 + 14z^3 - 8z^4 = (2z-1)\, z^2(1+4(1-z)),\\
P_4(z) &= z - 4z^2 + 5z^3 - 2z^4 = z(1-2z)(1-z)^2,\\
P_5(z) &= z^2 - 3z^3 + 2z^4 = z^2(1-2z)(1-z).
\end{aligned}
\tag{2.2}
$$

## 3. Proof of Theorem 1

The proof of Theorem 1 depends on the following representation of $s(x)$. Let $t = (x - x_i)/(x_{i+1} - x_i)$, $h_i = x_{i+1} - x_i$, $0 \leqslant t \leqslant 1$. Also we denote by $s_i(x)$ the restriction of the spline $s(x)$ to $[x_i, x_{i+1}]$. On using the representation (2.1), (2.2) we may express

$$
\begin{aligned}
s_i(x) = f(x_i)\, P_1(t) + f(z_{i+1})\, P_2(t) + f(x_{i+1})\, P_3(t)\\
+ h_i s_i'(x_i)\, P_4(t) + h_i s_i'(x_{i+1})\, P_5(t).
\end{aligned}
\tag{3.1}
$$

On account of (2.1) and the conditions

$$
s'(x_0) = f'(x_0), \qquad s'(x_k) = f'(x_k) \tag{3.2}
$$

we see that $s_i(x)$ given by (3.1) does satisfy (1.3) and that it is quartic in $[x_i, x_{i+1}]$ for $i = 0, 1, ..., k-1$. Moreover $s(x) \in C^1[0, 1]$ no matter how we choose $s'(x_i)$ $(i = 1, 2, ..., k-1)$. In order that $s(x) \in C^2[0, 1]$, we choose $s'(x_i)$ so that

$$
s''(x_i+) = s''(x_i-), \qquad i = 1, 2, ..., k-1. \tag{3.3}
$$

Equation (3.3) is equivalent to the following system of equations:

$$
\begin{aligned}
&-h_i s'(x_{i-1}) + 4(h_i + h_{i-1})\, s'(x_i) - h_{i-1} s'(x_{i+1})\\
&\quad = -11\left(\frac{h_{i-1}}{h_i} - \frac{h_i}{h_{i-1}}\right) f(x_i) + 16\left(\frac{h_{i-1}}{h_i} f(z_{i+1}) - \frac{h_i}{h_{i-1}} f(z_i)\right)\\
&\qquad - 5\left(\frac{h_{i-1}}{h_i} f(x_{i+1}) - \frac{h_i}{h_{i-1}} f(x_{i-1})\right).
\end{aligned}
\tag{3.4}
$$

But (3.4) is a strictly tri-diagonal dominant system. As is well known this system of equations has a unique solution.

## 4. Proof of Theorem 2

Our method of proof is similar to that of Hall and Meyer [6]. Let $f \in C^5[0, 1]$ and denote by $L_i[f, x]$ the unique quartic agreeing with $f(x_i)$, $f(x_{i+1})$, $f(z_{i+1})$, $f'(x_i)$, and $f'(x_{i+1})$. Let $s(x)$ be the twice continuously differentiable quartic spline function satisfying the conditions of Theorem 1. Then for $x_i \leqslant x \leqslant x_{i+1}$ we have

$$|f(x) - s(x)| \equiv |f(x) - s_i(x)| \leqslant |f(x) - L_i[f, x]| + |L_i[f, x] - s_i[x]|. \quad (4.1)$$

We now proceed by obtaining pointwise bounds of both terms on the right-hand side of (4.1). By a well-known theorem of Cauchy [5] we know that

$$|f(x) - L_i[f, x]| \leqslant \frac{h_i^5}{5!} \left| t^2 \left( \frac{1}{2} - t \right) (1 - t)^2 \right| U, \quad (4.2)$$

where

$$t = \frac{x - x_i}{h_i}, \qquad U = \max_{0 \leqslant x \leqslant 1} |f^{(5)}(x)|.$$

We next turn our attention to deriving a similar bound for $|L_i[f, x] - s_i(x)|$. From (2.1) we have

$$L_i[f, x] - s_i(x) = h_i[f'(x_i) - s_i'(x_i)]\, P_4(t) + h_i[f'(x_{i+1}) - s'(x_{i+1})]\, P_5(t). \quad (4.3)$$

We set

$$e'(x_i) = f'(x_i) - s_i'(x_i) \quad (4.4)$$

and we then have from (4.3)

$$|L_i(f, x) - s_i(x)| \leqslant h_i\, |e'(x_i)|\, |P_4(t)| + h_i\, |e'(x_{i+1})|\, |P_5(t)|. \quad (4.5)$$

As $P_4(t) = t(1 - 2t)(1 - t)^2$ and $P_5(t) = t^2(1 - 2t)(1 - t)$ are both positive for $0 \leqslant t \leqslant \frac{1}{2}$, and both negative for $\frac{1}{2} < t \leqslant 1$ it follows that

$$|P_4(t)| + |P_5(t)| = |P_4(t) + P_5(t)| = |t(1 - t)(1 - 2t)|. \quad (4.6)$$

Now, on using (4.5) and (4.6) it follows that

$$|L_i[f, x] - s(x)| \leqslant h_i \max\{|e'(x_i)|, |e'(x_{i+1})|\}\, |t(1 - t)(1 - 2t)|. \quad (4.7)$$

Next, we set

$$|e'(x_j)| = \max_{i=1,2,\ldots,k-1} |e'(x_i)|, \qquad h = \max_{i=0,2,\ldots,k-1} h_i. \tag{4.8}$$

Then, we may express (4.7) as

$$|L_i[f, x] - s(x)| \leqslant h\, |e'(x_j)|\, |t(1-t)(1-2t)|. \tag{4.9}$$

The next task is to find the upper bound for $|e'(x_j)|$. From (3.5) it follows that

$$h_j e'(x_{j-1}) - 4(h_j + h_{j-1})\, e'(x_j) + h_{j-1}\, e'(x_{j+1}) = B_0(f), \tag{4.10}$$

where

$$\begin{aligned} B_0(f) = {} & h_j f'(x_{j-1}) - 4(h_j + h_{j-1}) f'(x_j) + h_{j-1} f'(x_{j+1}) \\ & - 11\left[\frac{h_{j-1}}{h_j} - \frac{h_j}{h_{j-1}}\right] f(x_j) \\ & + 16\left[\frac{h_{j-1}}{h_j} f(z_{j+1}) - \frac{h_j}{h_{j-1}} f(z_j)\right] \\ & - 5\left[\frac{h_{j-1}}{h_j} f(x_{j+1}) - \frac{h_j}{h_{j-1}} f(x_{j-1})\right]. \end{aligned} \tag{4.11}$$

As $B_0(f)$ is a linear functional which is zero for polynomials of degree 4 or less, we can apply the Peano theorem [5] to obtain

$$B_0(f) = \int_{x_{j-1}}^{x_{j+1}} \frac{f^{(5)}(y)}{4!} B_0[(x-y)_+^4]\, dy. \tag{4.12}$$

From (4.12) it follows that

$$|B_0(f)| \leqslant \frac{1}{4!} U \int_{x_{j-1}}^{x_{j+1}} |B_0[(x-y)_+^4]|\, dy, \tag{4.13}$$

where $U = \max_{0 \leqslant x \leqslant 1} |f^{(5)}(x)|$. Next, we note from (4.12) that for $x_{j-1} \leqslant y \leqslant x_{j+1}$

$$\begin{aligned} B_0[(x-y)_+^4] = {} & -16(h_j + h_{j-1})(x_j - y)_+^3 \\ & + 4h_{j-1}(x_{j+1} - y)^3 - 11\left[\frac{h_{j-1}}{h_j} - \frac{h_j}{h_{j-1}}\right](x_j - y)_+^4 \\ & + 16\left[\frac{h_{j-1}}{h_j}(z_{j+1} - y)_+^4 - \frac{h_j}{h_{j-1}}(z_j - y)_+^4\right] \\ & - 5\,\frac{h_{j-1}}{h_j}(x_{j+1} - y)^4. \end{aligned}$$

In order to evaluate the integral of the rhs of (4.13) we rewrite the above expression in a form which shows its symmetry about $x_j$. Thus we have

$$\begin{aligned} B_0[(x-y)_+^4] &= \frac{h_j}{h_{j-1}}(-5(x_j-y)+h_{j-1})[(x_j-y)-h_{j-1}]^3, && x_{j-1}\leqslant y\leqslant z_j,\\ &= \frac{h_j}{h_{j-1}}(x_j-y)^2[11(x_j-y)^2-16h_{j-1}(x_j-y)+6h_{j-1}^2], && z_j\leqslant y\leqslant x_j,\\ &= \frac{h_{j-1}}{h_j}(x_j-y)^2[11(x_j-y)^2+16h_j(x_j-y)+6h_j^2], && x_j\leqslant y\leqslant z_{j+1},\\ &= \frac{h_{j-1}}{h_j}[-5(x_j-y)-h_j][(x_j-y)+h_j]^3, && z_{j+1}\leqslant y\leqslant x_{j+1}. \end{aligned} \tag{4.14}$$

From the above expression it follows that $B_0[(x-y)_+^4]$ is non-negative for $x_{j-1}\leqslant y\leqslant x_{j+1}$. Therefore it follows that

$$\int_{x_{j-1}}^{x_{j+1}} |B_0[(x-y)_+^4]|\,dy = \frac{h_jh_{j-1}[h_{j-1}^3+h_j^3]}{10}. \tag{4.15}$$

Now (4.13) and (4.15) give us

$$B_0(f)\leqslant \frac{Uh_jh_{j-1}[h_{j-1}^3+h_j^3]}{2(5!)}. \tag{4.16}$$

From (4.10), (4.8), and (4.16) we have

$$\max_{i=1,2,\ldots,k-1} |e'(x_i)| = |e_j'| \leqslant \frac{1}{6!}\frac{Uh_jh_{j-1}[h_{j-1}^3+h_j^3]}{(h_j+h_{j-1})}. \tag{4.17}$$

On combining (4.2), (4.7), (4.17) we have

$$\begin{aligned} |f(x)-s(x)| &\leqslant |f(x)-L_i[f,x]| + |L_i[f,x]-s_i(x)|\\ &\leqslant \frac{h^5}{5!}\left|t^2\left(\frac{1}{2}-t\right)(1-t)^2\right| U + h_i\,|e'(t_i)|\,|t(1-t)(1-2t)|\\ &\leqslant \frac{h^5}{5!}\left|t^2\left(\frac{1}{2}-t\right)(1-t)^2\right| U\\ &\quad + h_i\frac{1}{6!}\frac{Uh_jh_{j-1}}{(h_j+h_{j-1})}(h_{j-1}^3+h_j^3)\,|t(1-t)(1-2t)|\\ &\leqslant \frac{h^5}{5!}U\,|c(t)|, \end{aligned} \tag{4.18}$$

where

$$|c(t)| = \frac{|3t^2(1-t)^2(1-2t)| + |t(1-2t)(1-t)|}{6}$$

and

$$c(t) = \frac{t(1-2t)(1-t)(1+3t(1-t))}{6}.$$

This proves (1.4). Inequality (1.7) is a direct consequence of (4.17).

Now, we turn to prove that inequality (1.4) is best possible in the limit. Let $f_0(x) = x^5/5!$. Then from the Cauchy formula we have ($i = 0, 1, ..., k-1$)

$$\frac{x^5}{5!} - L_i\left[\frac{t^5}{5!}, x\right] = h^5 \frac{(1-t)^2(t-1/2)t^2}{5!}. \tag{4.19}$$

Furthermore, for equally spaced knots, we have

$$B_0\left(\frac{x^5}{5!}\right) = \frac{h^4}{5!} = e'(x_{i-1}) - 8e'(x_i) + e'(x_{i+1}). \tag{4.20}$$

Suppose for a moment that

$$e'(x_i) = \frac{-h^4}{6!} = e'(x_{i+1}) = e'(x_{i-1}). \tag{4.21}$$

Then on using (4.3) we have

$$\begin{aligned} L_i[f, x] - s(x) &= \frac{-h^5}{6!}(P_4(t) + P_5(t)) \\ &= \frac{h^5 t(1-t)(2t-1)}{6!}. \end{aligned} \tag{4.22}$$

On combining (4.19) and (4.22) we have

$$f(x) - s(x) = \frac{h^5}{5!}\left\{\frac{t(1-t)(t-1/2)}{3} + t^2(1-t)^2(t-1/2)\right\},$$
$$x_i \leqslant x \leqslant x_{i+1}. \tag{4.23}$$

From (4.23) follows that (1.4) is best possible provided we could prove that

$$e'(x_{i-1}) = e'(x_{i+1}) = e'(x_i) = \frac{-h^4}{6!}. \tag{4.24}$$

In fact (4.24) is attained only in the limit. The difficulty is the boundary conditions $e'(x_0)=e'(x_k)=0$. We can show, however, that as one moves many subintervals away from the boundaries, $e'(x_i)\to -h^4/6!$. We wish to apply (4.20) inductively to move away from the end conditions $e'(0)=e'(1)=0$. In order to do so we first establish that $e'(x_i)\leqslant 0$ for $i=0, 1, ..., k$. We reason by contradiction.

Suppose for some $i$, $i=1, 2, ..., k-1$, $e'(x_i)>0$. Then on using (1.7) we have

$$\frac{h^4}{6!}\geqslant \max |e'(x_j)| \geqslant \frac{1}{2}(e'(x_{i-1})+e'(x_{i+1})), \qquad j=1, 2, ..., k-1,$$

$$>\frac{1}{2}(e'(x_{i-1})-8e'(x_i)+e'(x_{i+1}))$$

$$=\frac{h^4}{2(5!)}$$

which is an obvious contradiction. Thus we just proved that $e'(x_i)\leqslant 0$ for $i=0, 1, ..., k$.

From (4.20) we have

$$B_0\left(\frac{x^5}{5!}\right)=\frac{h^4}{5!}=e'(x_{i-1})-8e'(x_i)+e'(x_{i+1}).$$

Therefore,

$$8e'(x_i)=\frac{-h^4}{5!}+e'(x_{i-1})+e'(x_{i+1})\leqslant\frac{-h^4}{5!}, \qquad i=1, 2, ..., k-1.$$

Hence we obtain

$$e'(x_i)\leqslant\frac{-h^4}{8(5!)}, \qquad i=1, 2, .., k-1. \tag{4.25}$$

Next, application of (4.20) and (4.24) gives

$$e'(x_i)\leqslant\frac{-h^4}{8(5!)}\left(1+\frac{1}{4}\right), \qquad i=2, ..., k-2.$$

Thus, it is clear that repeated use of (4.20) leads us to

$$e'(x_i)\leqslant\frac{-h^4}{8(5!)}\left(+\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^{l-1}}\right). \tag{4.26}$$

But the rhs of (4.25) $\rightarrow -h^4/6!$. Thus in the limit we have

$$e'(x_i) \leqslant \frac{-h^4}{6!}. \tag{4.27}$$

Earlier in (1.7) we proved that

$$|e'(x_i)| \leqslant \frac{h^4}{6!}; \tag{4.28}$$

(4.26) and (4.27) imply $e'(x_i) \rightarrow -h^4/6!$ corresponding to $f(x) = x^5/5!$ and for equally spaced knots in the limit. This proves Theorem 2 completely.

## 5. Conclusion

Theorem 2 demonstrates that the quartic $C^{(2)}$ spline gives the best possible order of approximation to functions from the smooth class $C^{(5)}$. We next discuss interpolation to the much less smooth class of functions which are merely continuous on $[0, 1]$. As $f'(0)$ and $f'(1)$ are not necessarily defined, we consider the quartic $C^{(2)}$ spline satisfying (1.1) and (1.3) but with boundary conditions $s'(0) = s'(1) = 0$. Now we state here without proof the following

Theorem 3. *Let $f \in C[0, 1]$. If $\{x_i\}_{i=0}^k$ is the partition of equally spaced knots, then for $x_i \leqslant x \leqslant z_{i+1} = (x_i + x_{i+1})/2$ and $t = (x - x_i)/h_i$, $i = 0, 1, ..., k-1$, we have*

$$|f(x) - s(x)| \leqslant c_1(1-t)\, \omega(f, h) \leqslant c_2 \omega(f, h), \tag{5.1}$$

*where*

$$c_1(t) = \left(1 + \frac{13}{3}t - 3t^2 - \frac{58}{3}t^3 + 16t^4\right)$$

$$c_2 = \max(c_1(t)) \approx 1.6572, \qquad 0 \leqslant t \leqslant \tfrac{1}{2}.$$

We finally conclude by raising the following problem: What is the smallest possible constant $c_1$ in (1.8)? Perhaps $c_1 = 1$.

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